

Self-similar solutions of the G-equation - analytic description of the flame surface

I. F. Barna

KFKI Atomic Energy Research Institute of the Hungarian Academy of Sciences,
(KFKI-AEKI), H-1525 Budapest, P.O. Box 49, Hungary

(Dated: October 28, 2010)

The main feature of the flame kinematics can be described with the G-equation. We investigate the solutions of the G-equation with the well-known self-similar Ansatz. The results are discussed and the method how to get self-similar solutions is briefly mentioned.

To understand the dynamics of flame fronts is a general interest for combustion engineering. One way to gain such knowledge is the investigation of the G-equation model. Since the G-equation was introduced by Markstein in 1964 [1] numerous analytical [2] and numerical approaches [3–5] have been used to find the solutions. The study of [2] uses the method of characteristics for nonlinear first order partial-differential equations(PDA).

Dekena [5] uses a computational fluid dynamics(CFD) code Fire to investigate turbulent flame propagation in Gasoline Direct Injection engines. Further references for application of the G-equation is given therein.

In the G-equation model the flame itself is treated as a surface (flame front) that separates the burnt from the unburnt gas. A detailed derivation of the model can be found in [2]. The flame front is described here by the relation of

$$G(r, z, t) = z - \zeta(r, t) = 0. \quad (1)$$

The most general form of the G-equation is the following:

$$\frac{\partial \zeta(r, t)}{\partial t} + u \frac{\partial \zeta(r, t)}{\partial r} - v + S_L \sqrt{1 + \left(\frac{\partial \zeta(r, t)}{\partial r} \right)^2} = 0. \quad (2)$$

Where $u(r, z, t)$ and $v(r, z, t)$ are the radial and the axial components of the gas velocity, and S_L is the constant modulus of the laminar burning velocity, respectively.

In the following we use a well-known method, which is very popular to investigate nonlinear PDAs, namely searching for self-similar solutions. We are looking for solution of (2) in the form of

$$\zeta(r, t) = t^{-\alpha} f\left(\frac{r}{t^\beta}\right) := t^{-\alpha} f(\eta). \quad (3)$$

The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $\zeta(x, t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on. This kind of solutions have a broad range of applicability, e.g. a large number of diffusion-reaction-type equations [6] or time-dependent telegraph-type heat-conduction equation [7] can be examined as well.

Substituting this into (2) we have

$$-\alpha t^{-\alpha-1} f(\eta) - \beta t^{-\alpha-\beta-1} f'(\eta) r - u(r, z, t) t^{-\alpha-\beta} f'(\eta) - v(r, z, t) + S_L \sqrt{1 + t^{-2\alpha-2\beta} [f'(\eta)]^2} = 0 \quad (4)$$

where prime denotes differentiation with respect to η .

After setting set $u = S_L = 1$ and $v = 0$, one can see that this is a non-linear ordinary differential equation(ODE) if and only if $\alpha = -1$ and $\beta = 1$ (*the universality relation*). The corresponding ODE we shall deal with is

$$f(\eta) - \eta f'(\eta) - f'(\eta) + \sqrt{1 + [f'(\eta)]^2} = 0. \quad (5)$$

The solutions can be easily obtained and read:

$$f(\eta)_1 = c_1 \eta - c_1 - \sqrt{1 + c_1^2}; \quad f(\eta)_2 = c_1 \sqrt{\eta(\eta - 2)} \quad (6)$$

where c_1 is the real integration constant. The solutions can be seen in Fig 1. The corresponding self-similar solutions are

$$\zeta(r, t)_1 = c_1 r - c_1 - \sqrt{1 + c_1^2}; \quad \zeta(r, t)_2 = c_1 t \sqrt{\left(\frac{r(r - 2t)}{t^2} \right)}. \quad (7)$$

Both solutions can be proven with direct derivation. The first solution is trivial however the non-trivial second one is presented on Fig. 2.

We may consider more sophisticated flow systems where the radial and axial components of the gas velocity and the S_L are functions of time and radial position. First let's consider that the velocities and the constant modulus of the laminar burning velocity are real numbers ($u, v, S_L \in \mathbb{R}$.) Now the ODE has the following form

$$f(\eta) - \eta f'(\eta) - u f'(\eta) - v + S_L \sqrt{1 + [f'(\eta)]^2} = 0. \quad (8)$$

The solutions became a bit more complicated

$$f(\eta)_1 = c_1 \eta + u c_1 + v - S_L \sqrt{1 + c_1^2}; \quad f(\eta)_2 = c_1 \sqrt{-u^2 - \eta^2 + S_L^2 - 2\eta u} + v. \quad (9)$$

The function $f(\eta)_2$ is presented in Fig. 3. with the following fixed set of parameters ($u = 2, v = -1, c_1 = 1, S_L = 3$). Note, that the solution has now a compact support with a non-vanishing first derivatives at the border. This means that there is no flux-conservation at the boundaries. There are more analytical solutions for special $u(x, t), v(x, t)$ and $S_L(x, t)$ functions. The only remaining task is to give reasonable physical interpretation for $u(x, t), v(x, t)$ and $S_L(x, t)$.

Another generally interesting question is the dispersion relation and the attenuation distances. It can be examined how wave equations or other nonlinear evolutionary equations propagate plain waves in time and in space. Inserting the usual plain wave approximation $\zeta(r, t) = e^{i(kr + \omega t)}$ into (2) the dispersion relation and the attenuation distance can be obtained. These are the followings

$$v_p = \frac{\omega}{Re(k)} = \frac{\omega}{\frac{1}{2} \sqrt{|-1 + k^2|} (1 - \text{signum}[-1 + k^2])}; \quad (10)$$

$$\alpha = \frac{1}{Im(k)} = \frac{1}{\omega + k + \frac{1}{2} \sqrt{|-1 + k^2|} (1 - \text{signum}[-1 + k^2])} \quad (11)$$

inserting the relation of $\omega = kc$ where c is the propagation velocity of the signal the formulas only depend on the wavenumber vector k . Considering $c = 1$ propagation speed Fig. 4. shows the phase velocity as the function of the wavenumber. The $(1 - \text{signum}[-1 + k^2])$ in the formula is responsible for the compact support of the function. Fig. 5. shows the attenuation distance of the various waves. At $k = 1$ the $\alpha(k)$ function is non-analytic.

In this short study we just wanted to present that self-similar solution can be easily used to generate analytic solutions for the G-equation. With more general, and more physical relations for the radial and axial gas flow velocity hopefully more physical solutions can be obtained. Anyhow, any kind of analytical solution of a nonlinear PDA can be useful giving a solid basis for testing complex and sophisticated two or three dimensional numerical finite-element computational fluid dynamics codes.

-
- [1] G. H. Markstein, *Nonsteady flame propagation* Pergamon Press, Oxford, 1964
 - [2] M.L. Bondar, R.M.M. Mattheij and J.H.M. ten Thije Boonkamp
 "Investigation of Bunsen flame dynamics by the method of characteristics"
IMA Journal of Applied Mathematics **72**, (2007) 405-419.
 M.L. Bondar and J.H.M. ten Thije Boonkamp
 "On the dynamics of the Bunsen flame" *Progress in industrial mathematics at ECMI 2004*, pages 207-211, 2005, (publisher Springer).
 M.L. Bondar, J.H.M. ten Thije Boonkamp and R.M.M. Mattheij
 "Theoretical modelling of flame-acoustic interaction"
Progress in industrial mathematics at ECMI 2006, pages 768-773, 2008, (publisher Springer).
 M.L. Bondar,
Acoustically Perturbed Bunsen Flames: Modelling, Analytical Investigations and Numerical Simulations,
PhD thesis, Eindhoven University of Technology 2007, ISBN 978-90-386-1118-1.
 - [3] H. Pitsch, "A G-equation formulation for large-eddy simulation of premixed turbulent combustion" *Center for Turbulence Research Annual Research Briefs 2002*.
 - [4] L. Liang, R. D. Reitz, J. Yi and C. O. Iyer, A G-equation Combustion Model Incorporating Detailed Chemical Kinetics for PFI/DI SI Engine Simulations.
 - [5] M. Dekema and N. Peters, "Combustion Modeling with the G-Equation" *Oil & Gas Science and Technology Rev. IFP*, **54**, (1999) 265-270.
 - [6] B.H. Gilding and R. Kersner, *Travelling Waves in Nonlinear Diffusion-Convection Reactions*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Verlag, Basel-Boston-Berlin, 2004, ISBN 3-7643-7071-8.
 - [7] I.F. Barna and R. Kersner "Heat conduction: a telegraph-type model with self-similar behavior of solutions" *J. Phys. A: Math. Theor.* **43**, (2010) 375210.

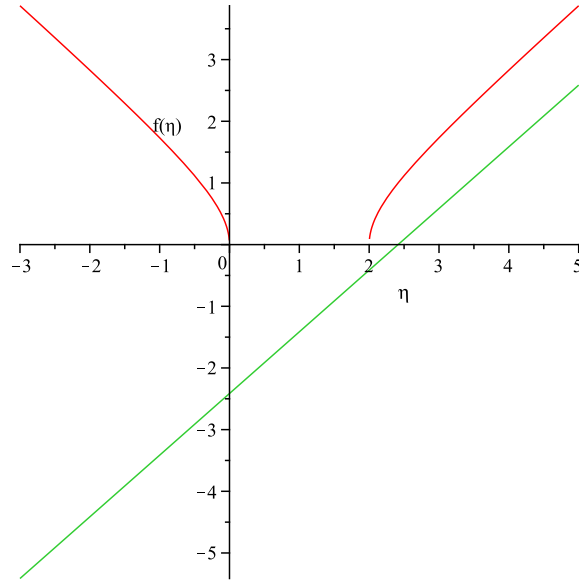


FIG. 1: Solutions of Eq. (5), red line presents $f(\eta)_2$ and the green one shows $f(\eta)_1$.

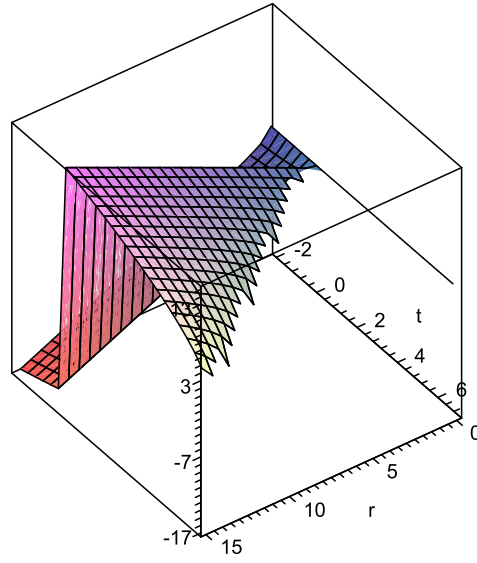


FIG. 2: The self-similar solution $\zeta(r, t)_2$, from Eq. (7) in the range $r = 0 \dots 15$, $t = -2 \dots 7$ for $c_1 = 1$.

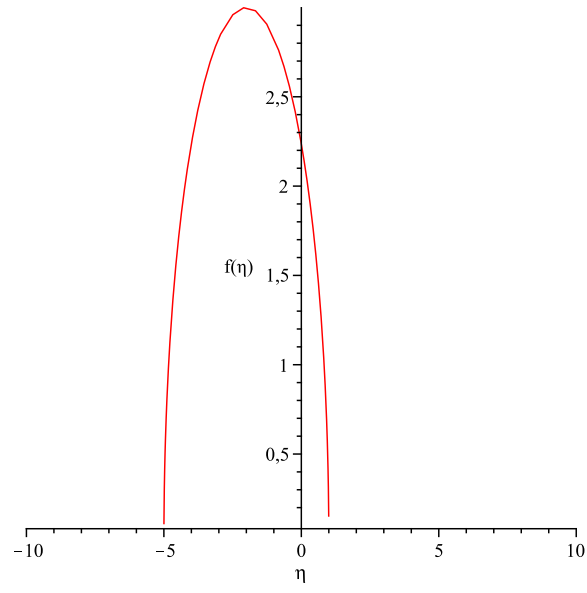


FIG. 3: The solution $f(\eta)_2$, $u = 2, v = -1, c_1 = 1, S_L = 3$.

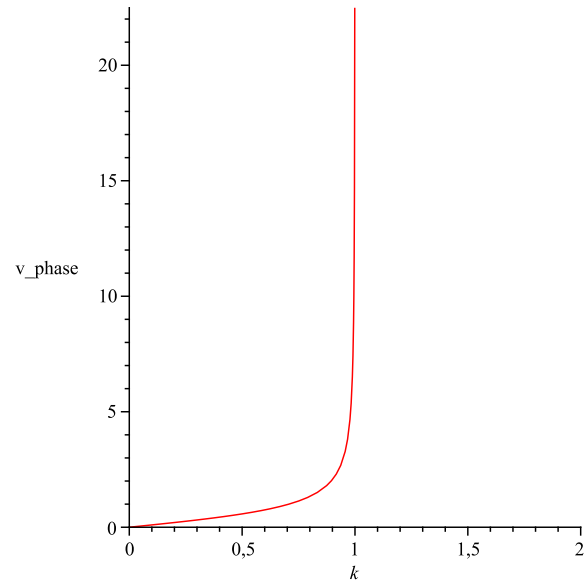


FIG. 4: The dispersion relation $v_p(k)$ for Eq. (2).

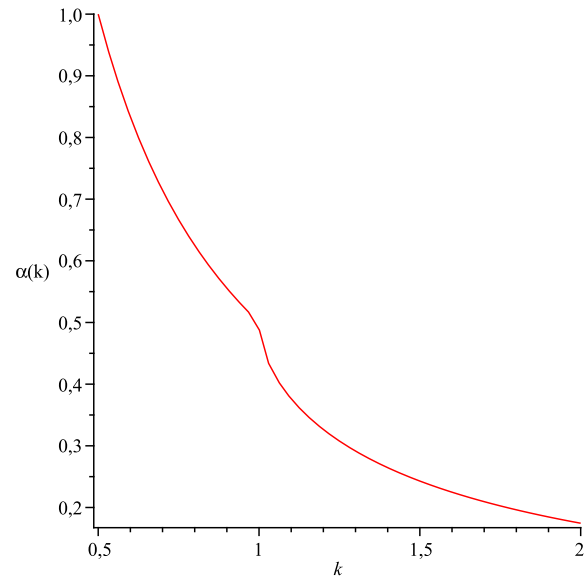


FIG. 5: The attenuation function $\alpha(k)$ for Eq. (2).